Name ____________________ Student Number ____________

All solutions are to be presented on the paper in the space provided. This exam is closed book.

(1) Find the following limits. Show all work. **1 Mark each**

(a) $\lim_{x\to 2} \dfrac{3x^2-5x-2}{x^2-3x+2}$

$$= \lim_{x\to 2} \frac{(x-2)(3x+1)}{(x-2)(x-1)}$$
$$= \lim_{x\to 2} \frac{3x+1}{x-1}$$
$$= 7$$

(b) $\lim_{x\to \frac{\pi}{2}} \cot x$

$$= \lim_{x\to \frac{\pi}{2}} \frac{\cos x}{\sin x}$$
$$= 0$$

Over→

(c) $\lim_{x \to 1} f(x)$, where

$$f(x) = \begin{cases} x^2 + 1, & x \leq 1 \\ \frac{2(x+2)}{x+1}, & x > 1 \end{cases}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1) = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{2(x+2)}{x+1} = 3$$

Since the left and right limits are different, the limit does not exist.

(d) $\displaystyle \lim_{x \to -1} \frac{\sqrt{x^2+3} - 2}{x+1}$

$$= \lim_{x \to -1} \left(\frac{\sqrt{x^2+3} - 2}{x+1} \right) \left(\frac{\sqrt{x^2+3} + 2}{\sqrt{x^2+3} + 2} \right)$$

$$= \lim_{x \to -1} \frac{x^2 + 3 - 4}{(x+1)\sqrt{x^2+3} + 1}$$

$$= \lim_{x \to -1} \frac{(x-1)(x+1)}{(x+1)(\sqrt{x^2+3} + 2)}$$

$$= \lim_{x \to -1} \frac{x-1}{\sqrt{x^2+3} + 2}$$

$$= -\frac{1}{2}$$

(e) $\displaystyle \lim_{x \to 0} \frac{\sin(\pi x)}{x}$

$$= \lim_{x \to 0} \frac{\sin(\pi x)}{\pi x} \pi$$

$$= \pi \lim_{x \to 0} \frac{\sin(\pi x)}{\pi x}$$

$$= \pi$$

Over→

(f) $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right)$

$$\begin{aligned} &= \lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{x}\right) \\ &= 0 \end{aligned}$$

(g) $\lim_{x\to\infty} e^{-x}$

$$= 0$$

(h) $\lim_{x\to 0^+} \log_4 x$

$$= -\infty$$

(i) $\lim_{x\to -\infty} \tan^{-1} x$

$$= -\frac{\pi}{2}$$

(j) $\lim_{x\to \frac{\pi}{2}^-} \sec x$

$$\begin{aligned} &= \lim_{x\to \frac{\pi}{2}^-} \frac{1}{\cos x} \\ &= \infty \end{aligned}$$

(2) Use first principles to find $f'(2)$ when $f(x) = \dfrac{1}{2x+1}$. **5 Marks**

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{\frac{1}{2(2+h)+1} - \frac{1}{2\cdot 2+1}}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{5+2h} - \frac{1}{5}\right) \end{aligned}$$

Over→

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{5-(5+2h)}{5(5+2h)}\right)$$
$$= \lim_{h\to 0} \left(\frac{-2h}{5(5+2h)}\right)$$
$$= \lim_{h\to 0} \frac{-2}{25+10h}$$
$$= -\frac{2}{25}$$

2 Marks each

(3) Find the derivatives of the following functions:

(a) $f(x) = x^n$

$$f'(x) = nx^{n-1}$$

(b) $f(x) = n^x$

$$f'(x) = n^x \ln(n)$$

(c) $f(x) = \csc x$

$$f'(x) = -\csc x \cot x$$

(d) $f(x) = \dfrac{g(x)}{h(x)}$

$$f'(x) = \frac{g'(x)h(x) - h'(x)g(x)}{[h(x)]^2}$$

(e) $f(x) = g(x)h(x)$

$$f'(x) = g'(x)h(x) + g(x)h'(x)$$

Over→

(4) Find the derivative of the following functions. Do not simplify. **2 Marks each**

(a) $f(x) = xe^x$

$$f'(x) = e^x + xe^x$$

(b) $f(x) = (\sin x)(\cos x)$

$$\begin{aligned} f'(x) &= \cos x \cos x - \sin x \sin x \\ &= \cos^2 x - \sin^2 x \end{aligned}$$

(c) $f(x) = \dfrac{\sqrt[3]{x}\tan x}{4^x}$

$$f'(x) = \frac{(\frac{1}{3}x^{-\frac{2}{3}}\tan x + \sqrt[3]{x}\sec^2 x)4^x - \sqrt[3]{x}\tan x 4^x \ln 4}{(4^x)^2}$$

(d) $f(x) = x2^x \sec x$

$$f'(x) = 2^x \sec x + x2^x \ln 2 \sec x + x2^x \sec x \tan x$$

(e) $f(x) = \dfrac{C}{x^2}$

$$f'(x) = \frac{-2C}{x^3}$$

(5) Find the equation of the tangent line to $f(x) = xe^x$ at $x = 1$. **5 Marks**
Using the derivative from question 4a, we have $f'(1) = 2e$. Also $f(1) = e$. So the equation of the tangent line is

$$\begin{aligned} y - f(1) &= f'(1)(x-1) \\ y &= 2e^x - e \end{aligned}$$

Over→

(6) Answer the following questions about the function

$$f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$$

3 Marks (a) Show that $f(x)$ is not continuous at $x = 1$

$$\begin{aligned} \lim_{x\to 1} \frac{x^2-1}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} \\ &= \lim_{x\to 1}(x+1) \\ &= 2 \end{aligned}$$

Since $f(1) = 3$, $\lim_{x\to 1} f(x) \neq f(1)$, so $f(x)$ is not continuous at $x = 1$.

2 Marks (b) Redefine $f(x)$ at $x = 1$ so that it is continuous everywhere.
Define $g(x) = x + 1$. Then $g(x) = f(x)$ when $x \neq 1$ and $g(1) = 2$.

5 Marks (7) Evaluate $\lim_{x\to 1} x^2 e^{\sin(\frac{\pi}{2}x)}$. This is a one line problem. Be sure to state what essential property you are using to evaluate the limit.
Since x^2,e^x and $\sin x$ are all continuous, so is $x^2 e^{\sin(\frac{\pi}{2}x)}$. Therefore, using continuity,

$$\lim_{x\to 1} x^2 e^{\sin(\frac{\pi}{2}x)} = 1^2 e^{\sin(\frac{\pi}{2}1)} = e$$

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